Hauptseminar: Active Matter
Stokes Flow and Life at Low Reynolds Numbers

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1 Life at low Reynolds number

The motion of an incrompressible Newtonian viscous fluid is described with the Navier Stokes equation:

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v} \right) = -\underline{\nabla} p + \eta \nabla^2 \underline{v} + \underline{F} \tag{1}$$

where ρ is the density, $\underline{v}$ the velocity, η the viscosity and p the pressure of the fluid and $\underline{F}$ are external forces. The Reynolds number comes naturally from non dimensionalizing the Navier Stokes equation, here without external forces:

$$Re \left(\frac{\partial \underline{v}'}{\partial t'} + (\underline{v}' \cdot \underline{\nabla})\underline{v}' \right) = -\underline{\nabla} p' + \nabla^2 \underline{v}' \tag{2}$$

with $\underline{v}' = \underline{v}/U$, $t' = t/(U/l)$, $p' = pl/(\eta U)$ and the characteristic lenght l and velocity U. The Reynolds number is defined as

$$Re = \frac{lU\rho}{\eta} \tag{3}$$

Since the number depends on the size and the velocity of an object it is clearly that different swimmers in the same fluid have different Reynolds numbers. For example, a human being ($l \approx 1.7$ m) who swims in water ($\rho = 1000$ kg/m^3, $\eta \approx 10^{-3}$Pa·s) at a constant velocity $U \approx 1$ m/s has a Reynolds number of $\approx 10^6$. In comparison, a fish ($l \approx 10^{-1}$ m) which also swims in water at a velocity $U \approx 0.1$ m/s has a Reynolds number of $\approx 10^4$. For bacteria in water ($l \approx 10^{-6}$ m, $U \approx 10^{-6}$ m/s), the Reynolds number decreases to $\approx 10^{-5}$. In order to tune the Reynolds numbers we can, for example, reduce the size of an object and/or its speed.
Next we will examine the effects of small Reynolds numbers. For this, we consider the Navier Stokes equation (eq. (2)) in which we can neglect the left terms which describe the inertia of a liquid. The equation then becomes the linear and time independent Stokes equation for incompressible fluids

$$0 = -\underline{\nabla} p' + \nabla^2 \underline{v}'. \tag{4}$$

The solution of eq. (4) is the so called Stokes flow. A consequence of the lack of inertia of the liquid is that a particle that has been dropped into the fluid instantaneously reaches its stationary velocity. Another consequence is that the relaxation of the liquid is also instantaneous. For a swimmer, this means that the reached swimming distance does not depend

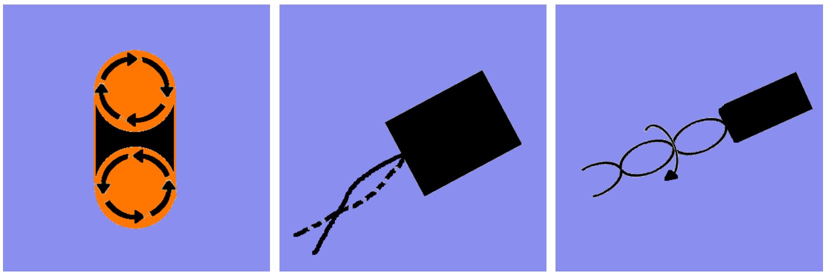

Figure 1: From left to right. A cut through view of a donut with rotating surface. A swimmer with a flexible oar and a swimmer with a rotating helix. All swimmer are non-reciprocal.

on the speed of its swimming movements but only on the type of its movements or more precisely its configuration. As an example we consider a scallop. This animal swimms by slowly opening its shell and then quickly closing it. During the opening of the shell, a force is generated by displacing the fluid, which instantly disappears when the shell is completely opened. At low Reynolds number, the same force is produced during the closing of the shell only in the opposite direction. So the scallop retraces its trajectory and returns where it started. Therefore, the scallop can not swim. The swimming movement of the scallop is invariant under time reversal. Such a movement is called reciprocal [1]. Purcell' "no scallop" theorem states that non-reciprocal motion, which breaks the time reversal symmetry is needed to produce a net displacement at low Reynolds number.

A simple non-reciprocal swimmer is a donut with rotating surface. Fig 1 (left) shows the cut of this donut. The arrows indicate the direction of rotation. Through the rotation, the liquid flows around the donut and through the donut hole. As long as the donut rotates, it floats in a line. An addional non-reciprocal drive is a flexible oar (fig 1, middle). The oar bends one way during its downmove and other way during its upmove. This satisfies the break of the time reversal symmetry. If the oar is stiff it would be a reciprocal drive. Since the movement of the oar is the same under time reversal. A swimmer with a rotating helix (fig. 1, right) breaks also the time reversal symmetry.

In the next section we will look at a simple man made non-reciprocal swimmer, namely three linked spheres.

2 Three linked spheres

A. Najafi and R. Golestanian suggested a simple and experimantally accessible model system that can swim in high viscous fluids [2]. This swimmer consists of three spheres with radius R (fig. 2). They are connected by rigid slender arms aligned along one direction e.g. the x direction [2]. In sphere number 1, two motors are fitted with which the length of the two arms are varied. In the initial state, sphere number 2 and 3 are equally distant to sphere number 1. One cycle is divided into four steps:

(a) In the first motion, the length of right arm is fixed and the length of the left arm is reduced by e

(b) In the second step, the length of the right arm is reduced by the same lenght e and the left arm is fixed.

(c) In the third step, the right arm is fixed and the length of the left arm is lengthened to L.

(d) In the last step, the right arm is lengthened to L and the left arm is fixed.

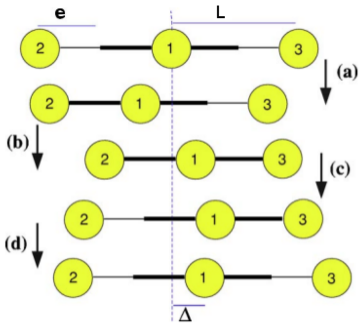

Figure 2: Complete cycle of the three linked spheres [2]

The lengthening and shortening of the arms occurs at a constant speed W. After one cycle, the swimmer has moved to the right by the distance Δ. But why does this swimmer swim? For this, we consider a particle in a liquid. When the liquid is at rest, the particle arccordingly rests. If the fluid flows, the particle experiences a force which causes the particle to move. A flow can be caused by a moving second particle. This flow is felt by the previously resting particle which begins to move. This generates a new flow which affects the second particle. It is therefore possible to say that each flow of a particle influences the motion of the other particles, and vice versa. If we now look at the three linked spheres without sphere 3 and reduce the distance between sphere 1 and 2, then both spheres generate a flow. The flow of sphere 1 influence sphere 2 and the flow of sphere 2 influences the movement of sphere 1. Since both spheres are the same, the generated flows are also the same, they differ only in the direction. This is why they both set the same distance back, namely $(l - e)/2$. If the distance is increased, both spheres move back to their starting positions. These two linked spheres can't swim at low Reynolds number.
Now we add the third sphere. During step (a), the sphere 1 and 3 produce a flow which sphere 2 experiences and sphere 2 generates a flow which acts on sphere 1 and 3. Since the summed flow of sphere 1 and 3 is greater than that of sphere 2, the swimmer moves in the negative x-direction.
In step (b), the flow generated by the movement of sphere 1 and 2 is greater than that of sphere 3. As a result, the swimmer moves in the positive x-direction. During the third step, the swimmer moves in the positive x-direction due to the occurring flows.
In the last step, the swimmer moves in the negative x-direction. Since the flow of sphere 1 and 2 is greater than that of sphere 3.
Note that the effects of the flow depend on the distance between the spheres. In order to be able to calculate the displacements of the swimmer after each step, the velocities of each sphere is needed. The velocities depend on the flows. To calculate the effect of the flow produced by e.g. sphere 3 on sphere 1 we use the lorentz Lorentz reciprocal theorem. This requires a known flow, for example, that of a moving sphere. With the aid of the Lorentz reciprocal theorem one can also determine the force and the torque which act on the particle through the flow. The resulting equations for force and torque are called Faxens laws. For each sphere, the no-slip boundary condition

$$\underline{u} = \underline{V_i} \tag{5}$$

must be fulfilled, where $\underline{u}$ is the velocity of fluid on the ith sphere and $\underline{V}_i$ is the velocity of the ith sphere. In addition, the velocity of the fluid $\underline{u}$ is zero at infinity. For two spheres interacting hydrodynamically, there is an exact solution ([3] ch. 6.2). For more than two spheres, there is no exact analytical solution which fulfilles the boundary conditions simultaneously ([3], ch. 6.1). An approximate solution is provided by the Rotne Prager tensor. In the following chapters, we shall derive the Lorentz reciprocal theorem, Faxens laws and the Rotne Prager tensor.

2.1 The Lorentz reciprocal theorem

With the Lorentz reciprocal theorem, it is possible to obtain informations about an unknown flow by means of a known flow. For this both flows must be in the same domain. The derivation follows chapter 3.5 in [3]. We start with the Stokes equation:

$$0 = -\underline{\nabla} p + \eta \nabla^2 \underline{v} \tag{6}$$

and rewrite it by introducing the stress tensor of a Newtonian fluid

$$\Pi_{ij} = -\delta_{ij} p + 2\eta \Delta_{ij} \quad \Delta_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{7}$$

where δ_{ij} is the Kronecker delta and Δ_{ij} is the viscous stress tensor. This tensor is symmetric $\Pi_{ij} = \Pi_{ji}$. The Stokes equation is then given by

$$\underline{\nabla} \cdot \underline{\underline{\Pi}} = \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \tag{8}$$

where we use the Einstein notation (repeated indices are summed). In order to derive the Lorentz theorem, we use the disappering divergence of the velocity field

$$\underline{\nabla} \cdot \underline{v} = 0, \tag{9}$$

which is a consequence of the incompressibility of the fluid. Now, we consider two motions (v, Π) and (v', Π') of the same fluid in the same domain. Both flows satisfey eq. (8) and eq. (9). The contraction of the stress tensor with the primed viscous stress tensor leads to

$$\Pi_{ij}\Delta'_{ij} = -p \underbrace{\partial_{x_i} v'_i}_{=0} + 2\eta \underbrace{\Delta_{ij}\Delta'_{ij}}_{=\Delta'_{ij}\Delta_{ij}} \tag{10}$$

$$\rightarrow \Pi_{ij}\Delta'_{ij} = \Pi'_{ij}\Delta_{ij} \tag{11}$$

and

$$\underline{\underline{\Pi}} : \underline{\underline{\Delta}}' = \Pi_{ij}\Delta'_{ij} = \underline{\nabla} \cdot (\underline{\underline{\Pi}} \cdot \underline{v}') \tag{12}$$

where we used the chain rule, the symmetry of Π and eq. (8). We obtain

$$\underline{\nabla} \cdot (\underline{\underline{\Pi}} \cdot \underline{v}') = \underline{\nabla} \cdot (\underline{\underline{\Pi}}' \cdot \underline{v}) \tag{13}$$

by using eq. (11). Multiplying both sides with the volume element dV and integrating over an arbitrary fliud volume V and using the divergence theorem we get

$$\int_S \left(\underline{\underline{\Pi}} \cdot \underline{v}'\right) \cdot \mathrm{d}\underline{s} = \int_S \left(\underline{\underline{\Pi}}' \cdot \underline{v}\right) \cdot \mathrm{d}\underline{s}, \tag{14}$$

the Lorentz reciprocal theorem. When a particle is in the fluid, then the surface S is composed of the surface of the particle and the surface at infinite.

2.2 Faxens laws

The force acting on a particle in a flow is described by Faxens first law. Faxens second law describes the torque exerted by an external flow on a particle. The derivation follows [4]. Now we use the reciprocal theorem to derive Faxens laws. For this purpose we consider the flow of a sphere in a flowing fluid. If we subtract the flow of the fluid without the sphere, we get the velocity disturbance caused by the sphere:

$$\underline{v} = \underline{u} - \underline{u}_a. \tag{15}$$

where $\underline{u}_a$ is the velocity of the ambient fluid without the sphere and $\underline{u}$ is the flow around the sphere. In order to use the Lorentz reciprocal theorem, we need two flows. The second known flow with known velocity disturbance is given by

$$\underline{v}' = \underline{u}' - \underline{u}'_a, \tag{16}$$

where $\underline{u}'$ is the fluid velocity around a sphere and $\underline{u}'_a$ is the known ambient flow. The corresponding stress tensors are given by

$$\underline{\underline{\sigma}} = \underline{\underline{\Pi}} - \underline{\underline{\Pi}}_a \tag{17}$$

$$\underline{\underline{\sigma}}' = \underline{\underline{\Pi}}' - \underline{\underline{\Pi}}'_a \tag{18}$$

where $\underline{\underline{\Pi}}'_a$ and $\underline{\underline{\Pi}}_a$ are the stress tensors of the known and unknown ambient flow. We assume that the velocity disturbances disappear at infinite. As a result, the integrals in eq. (14) pass over the surface of a sphere S_{sp}:

$$\int_{S_{sp}} \left(\underline{\underline{\sigma}} \cdot \underline{v}'\right) \cdot \mathrm{d}\underline{s} = \int_{S_{sp}} \left(\underline{\underline{\sigma}}' \cdot \underline{v}\right) \cdot \mathrm{d}\underline{s} \tag{19}$$

Plug in the equations (15) to (18) leads to

$$\int_{S_{sp}} \left(\underline{\underline{\sigma}} \cdot \underline{u}'_a\right) \cdot \mathrm{d}\underline{s} - \int_{S_{sp}} \left(\underline{\underline{\Pi}}_a \cdot \underline{v}\right) \cdot \mathrm{d}\underline{s} = \int_{S_{sp}} \left(\underline{\underline{\Pi}}' \cdot \underline{u}_a\right) \cdot \mathrm{d}\underline{s} - \int_{S_{sp}} \left(\underline{\underline{\Pi}}_a \cdot \underline{u}'\right) \cdot \mathrm{d}\underline{s} \tag{20}$$

In order to determine the force we use the uniform flow $\underline{u}'_a = \underline{U}$, $\underline{\underline{\Pi}}_a = 0$ past the partical and insert it in eq. 20:

$$\int\limits_S \underline{\underline{\sigma}} \cdot \underline{U} \, \mathrm{d}s = \int\limits_{S_{sp}} \left(\underline{\underline{\Pi}}' \cdot \underline{u}_a\right) \cdot \mathrm{d}\underline{s} - \int\limits_{S_{sp}} \left(\underline{\underline{\Pi}}_a \cdot \underline{u}'\right) \cdot \mathrm{d}\underline{s} \tag{21}$$

$$\rightarrow \underline{F} \cdot \underline{U} = \int\limits_{S_{sp}} \left(\underline{\underline{\Pi}}' \cdot \underline{u}_a\right) \cdot \mathrm{d}\underline{s} - \int\limits_{S_{sp}} \left(\underline{\underline{\Pi}}_a \cdot \underline{u}'\right) \cdot \mathrm{d}\underline{s} \tag{22}$$

with the drag force $\underline{F}$. As $\underline{u}'$, we use the velocity field of a solide sphere with radius a [3]:

$$u'_r = -\frac{1}{2} U \cos(\theta) \left(\frac{a}{r}\right)^2 \left(\frac{a}{r} - 3\frac{r}{a}\right) \quad \text{and} \quad u'_\theta = -\frac{1}{4} U \sin(\theta) \left(\left(\frac{a}{r}\right)^2 - 3\right) \tag{23}$$

where u_r is the radial component and u_θ is polar component of the velocity $\underline{u}'$. Further, we develop the velocity $\underline{u}_a$ in a Taylor series:

$$\underline{u}_a(\underline{r}) \approx \underline{u}_a(\underline{r}_0) + [\nabla \underline{u}_a(\underline{r}_0)] \cdot (\underline{r} - \underline{r}_0)... \tag{24}$$

where $\underline{r}_0$ is the position of the centre of the sphere. We set the centre of the sphere to the origin ($\underline{r}_0 = 0$). Using the equations (23) and (24) to solve the integral of eq. (22) leads to Faxens first law:

$$\underline{F} = 6\pi\eta a \left(\underline{u}_a(0) + \frac{1}{6} a^2 \nabla^2 \underline{u}_a(0)\right). \tag{25}$$

This equation describes the drag force exerted by the fluid on a solid sphere. To calculate Faxens second law we use a rotating uniform flow around a sphere with $\underline{u}_a = \underline{\Omega} \times \underline{r}$, where $\underline{\Omega}$ is the angular velocity. Eq. 20 is then given by

$$\underline{T} \cdot \underline{\Omega} = \int\limits_{S_{sp}} \left(\underline{\underline{\Pi}}' \cdot \underline{u}_a\right) \cdot \mathrm{d}\underline{s} \tag{26}$$

where $\underline{T}$ is the torque. As $\underline{u}'$, we use the rotating velocity field around a solide sphere [3]:

$$\underline{u}' = \frac{3}{2} a \frac{\underline{\Omega} \times \underline{r}}{r^3} \tag{27}$$

The calculation steps are the same as previously, and lead to Faxens second law:

$$\underline{T} = 4\pi\eta a^3 (\nabla \times \underline{u}_a(0)) \tag{28}$$

which describes the torque exerted on a solid sphere by a rotational flow.

2.3 Rotne Prager tensor

As mentioned above, there is no analytical solution which fulfill all boundary conditions simultaneously (no-slip condition eq. (5) for each particle) if more than two particles interact hydrodynamically with one another. Therefore, in this section, we derive the Rotne Prager tensor. To do this, we define the force vector $\underline{F} = (\underline{f}_0, ..., \underline{f}_N)$ and the velocity vector $\underline{V} = (\underline{v}_0, ..., \underline{v}_N)$, where $\underline{v}_i$ is the velocity of the ith particle. The force $\underline{f}_i$, is the force acting on the ith particle. This force is produced by the movements of the other particles. Due to the linearity of the Stokes equation (eq. (6)), the following equation applies to the velocities and forces

$$\underline{V} = \underline{\underline{D}} \cdot \underline{F} \tag{29}$$

where $\underline{\underline{D}}$ is the unknown moility tensor. In addition, the Stokes flow minimizes the energy dissipation which is defined by

$$\epsilon := \frac{1}{2\eta} \int\limits_V \underline{\underline{\Delta}} : \underline{\underline{\Delta}} \, \mathrm{d}^3 \underline{r}' \tag{30}$$

with the viscous stress $\underline{\underline{\Delta}}$. The energy dissipation can also be written as

$$\underline{V} \cdot \underline{F} = \left(\underline{\underline{D}} \cdot \underline{F} \right) \cdot \underline{F}. \tag{31}$$

If $\underline{\underline{\Delta}}$ is the exact viscous stress and $\underline{\underline{D}}$ is the exact mobility tensor, eq. (30) and eq. (31) must have the same energy dissipation value. Therefore, we found

$$\left(\underline{\underline{D}} \cdot \underline{F} \right) \cdot \underline{F} = \frac{1}{2\eta} \int\limits_V \underline{\underline{\Delta}} : \underline{\underline{\Delta}} \, \mathrm{d}^3 \underline{r}'. \tag{32}$$

To derive an approximation for the mobility tensor $\underline{\underline{D}}^*$ we replace the exact viscous stress in eq. (32) with a trial function distribution $\underline{\underline{\Delta}}^*$. Every energy dissipation with a trial function will be larger:

$$\left(\underline{\underline{D}} \cdot \underline{F} \right) \cdot \underline{F} \leq \frac{1}{2\eta} \int\limits_V \underline{\underline{\Delta}}^* : \underline{\underline{\Delta}}^* \, \mathrm{d}^3 \underline{r}'. \tag{33}$$

$\underline{\underline{\Delta}}^*$ must fulfill the conditions

$$\underline{\underline{\Delta}}^* = \left(\underline{\underline{\Delta}}^* \right)^T, \quad \mathrm{tr} \left(\underline{\underline{\Delta}}^* \right) = 0 \tag{34}$$

which ensures the incompressibility of the flow. The trial viscous stress should obey the force balance of each sphere

$$\int\limits_{S_i} \left(\underline{\underline{\Delta}}^* + p^* \underline{\underline{I}} \right) \cdot \, \mathrm{d}^2 \underline{r} = \underline{f}_i \tag{35}$$

where p^* is local pressure and $\underline{\underline{I}}$ is the three dimensional unit tensor. A simple trial stress distribution is generated by superposition of the individual contributions of each particle

$$\underline{\underline{\Delta}}^*(\underline{r}) = \sum_i \underline{\tau}_i (\underline{r} - \underline{r}_i') \tag{36}$$

with $\underline{r}^* = \underline{r} - \underline{r}_i'$ the position the particle. For a flow resting at infinity, the trial $\underline{\tau}_i(\underline{r}^*)$ is choosen as

$$\underline{\tau}_i(\underline{r}^*) = \eta \left(\frac{\partial \underline{u}_i}{\partial \underline{r}^*} + \left(\frac{\partial \underline{u}_i}{\partial \underline{r}^*} \right)^T \right) \tag{37}$$

with $\underline{\tau}_i(\underline{r}^*) = 0$ for $r^* < R$ and the flow $\underline{u}_i$ around a sphere of radius R subject to a force $\underline{f}_i$:

$$\underline{u}_i(\underline{r}^*) = \frac{1}{8\pi\eta R} \left(\left(\frac{R}{r^*} + \frac{1}{3} \frac{R^3}{r^{*3}} \right) \underline{f}_i + \left(\frac{R}{r^{*3}} - \frac{R^3}{r^{*5}} \right) \underline{f}_i \cdot \underline{r}^* \underline{r}^* \right). \tag{38}$$

Replacing the $\underline{\underline{D}}$ by $\underline{\underline{D}}^*$ and $\leq$ by $=$ in eq. (32), plug in eq. (36) to eq. (38) and solving the integral leads to the Rotne-Prager tensor [5]:

$$\underline{\underline{D}}^*_{ii} = \frac{1}{6\pi\eta R}\underline{\underline{I}} \tag{39}$$

$$\underline{\underline{D}}^*_{ij} = \frac{1}{8\pi\eta r^3_{ij}}\left(\left(\underline{\underline{I}}\, r^2_{ij} + \underline{r}_{ij}\underline{r}_{ij}\right) + \frac{2R^2}{r^2_{ij}}\cdot\left(\frac{1}{3}\underline{\underline{I}}r^2_{ij} - \underline{r}_{ij}\underline{r}_{ij}\right)\right) \quad i \neq j \tag{40}$$

with the relativ position $r_{ij} := r_i - r_j$ of the sphere i to the sphere j. The diagonal elements, $\underline{\underline{D}}^*_{ii}$ describe the sphere's response to an external force acting on it. The off diagonal elements, $\underline{\underline{D}}^*_{ij}$, describe the hydrodynamic interaction between the spheres.

2.4 Swimming of the three linked spheres

Lets go back to the three linked spheres. Now we can use the Rotne Prager tensor to calculate the velocity of each sphere in each step. The distance between the spheres is much larger than their radii ($r_{ij} >> R$), so the Rotne Prager tensor can be approximated by its lowest decaying components

$$\underline{\underline{D}}^*_{ii} = \frac{1}{6\pi\eta R}\underline{\underline{I}} \tag{41}$$

$$\underline{\underline{D}}^*_{ij} = \frac{1}{6\pi\eta R}\left(\frac{3R}{2r^3_{ij}}\underline{r}_{ij}\underline{r}_{ij} + \frac{3R}{4r_{ij}}\left(\underline{\underline{I}} - \frac{\underline{r}_{ij}\underline{r}_{ij}}{r^2_{ij}}\right)\right) \quad i \neq j \tag{42}$$

This tensor has only terms with order $1/r_{ij}$. One can derive the velocity $\underline{V}_i$ of the ith sphere with

$$\underline{V}_i = \sum_{j=1}^{3}\underline{\underline{D}}^*_{ij}\cdot\underline{F}_j \tag{43}$$

Neglecting the external forces, such as gravitation, leads to the condition of a force free system

$$\sum_{j=1}^{3}\underline{F}_j = 0 \tag{44}$$

The arms are moved in and out at a constant speed W. One cycle therefore takes $4e/W$. The average speed of the swimmer is

$$\underline{V}_s = \sum_{i=1}^{3}\frac{1}{3}\underline{V}_i = W\frac{\Delta}{4e}\underline{\hat{e}}_x \tag{45}$$

where $\underline{\hat{e}}_x$ is the unity vector in x-direction. Figure 3 shows the net displacement after one non-reciprocal cycle as function of the internal relative displacement e [2]. The net displacement Δ increases quadratically with the internal displacement e.

3 Collective locomotion of reciprocal swimmers

In this section we want investicate the behavior of two dimers (fig. 4) which can interact hydrodynamically. Each dimer consist of two spheres with radii a_1, a_2 and b_1, b_2 and have the lengths $\ell_a(t), \ell_b(t)$. The dimers are seperated by the distance $d(t)$. Each dimer is force

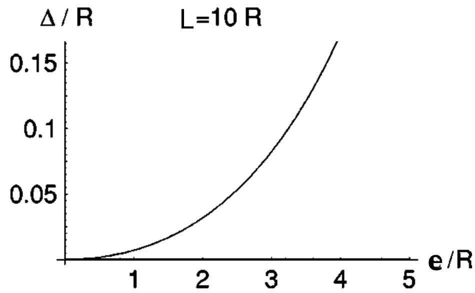

Figure 3: Dimensionless net displacement of the swimmer after on cycle as function of the relative displacement [2].

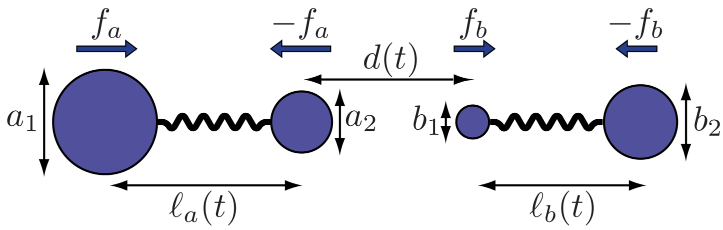

Figure 4: Two dimers each consist of two solid spheres with radii a_1, a_2 and b_1, b_2 and have the lengths $\ell_a(t), \ell_b(t)$, both dimers are separated by the length $d(t)$ [6].

free and for R the size of the body we consider that $R << \ell_i << d$, so we can use eq. 43 and eq. 44 to derive the velocities. First we look at a single dimer, for any periodic forces or deformation exerted on the dimer the average velocity over time is zero, $\langle v_a \rangle = 0$ [6] which satisfy the scallop theorem. Because the single dimer is a reciprocal swimmer.
Now we look at the behavior of two dimers (fig. 4). Therefore we define the average collectiv swimming speed

$$\langle V \rangle := \langle v_b + v_a \rangle / 2 \tag{46}$$

and the relativ swimming speed

$$\Delta V := \langle v_b - v_a \rangle \tag{47}$$

of the dimers. There are two possible motions, the force driven and displacement driven motion. For the force driven motion the internal forces f_i are specified and for the displacement driven motion the deformation $\ell_i(t)$ of each dimer is specified. In order to be able to calculate the dynamics of the four bodies, we do a Taylor expansion of $\underline{\underline{D}}^*$ (eq. (42)). and derive the average and the relative swimming speed for both motions [6]. The first non-vanishing term has order $1/d^3$

$$\langle V \rangle \propto \frac{1}{d^3} \quad \Delta V \propto \frac{1}{d^3} \tag{48}$$

where d is the distance between the dimers. The two dimers thus behave as a force quatropol. The swimming behavior depends on their relative orientation and the size of a_1, a_2, b_1 and b_2[6]. This results shows that the scallop theorem breaks for two reciprocal swimmers which interact hydrodynamically.

4 Conclusion

At low Reynolds number inertia plays no role. Therefore the Navier-Stokes equation becomes linear and time independent, so it dosen't matter if a swimmer change his shape fast or slowly. This fact is taken up in the "No Scallop" theorem. A reciprocal swimmer can't go anywhere. Reciprocal means that if the sequence of swimming movements does not change under time reversal. But it can swim if the set of motions breaks the time-reversal symmetry. A simple non-reciprocal swimmer are three linked spheres. This swimmer swimms due to the flow caused by the directed movement of the spheres. Each produced flow of a sphere is felt by the other ones. With the Lorentz reciprocal theorem it is possible to derive the flow produced by sphere 1 on sphere 2. Additionaly, with the aid of this theorem we can derive Faxen's laws. The first one describes the force exerted by an ambient flow on a solid sphere. Faxen's second law describes the torque exerted by a rotational flow. The Rotne Prager tensor was introduced to derive the hydrodynamical interaction between many particle.
The swimming of a single dimer was investigated. Due to his reciprocal motion it can't go anywhere. But if the dimer interacts hydrodynamically with a second dimer both can swim collectively and break the scallop theorem. The collective swimming behaviour depends on their relative orientions and of the size of each sphere.

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