Hydrodynamics of Newtonian Fluids

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Abstract

In this review we provide an overview of the hydrodynamics of viscous, Newtonian fluids, particularly for the case of flow at low Reynolds number. The case of low Reynolds number is most relevant for microswimmers and active matter. We begin with a general description of a fluid and the equations which a fluid obeys and narrow down to the description of an incompressible, Newtonian fluid at low Reynolds number. On the way, we discuss the so-called linear fluctuating hydrodynamics and the algebraic decay of the velocity correlation function. Once we have narrowed our discussion down to the low Reynolds number regime, we discuss some methods for solution of these linear equations. The variational method for hydrodynamics, the stream function, and the method of singularities are discussed.

I. Viscous Fluids

i. Mathematical Description of a Fluid

Fluid dynamics is the study of flows of fluid. However, before we begin discussing the dynamics of fluids we must define what we mean when we refer to a fluid. We know that a fluid is comprised of discrete particles that are in constant motion and can collide with each other and boundary walls. For our purposes, we would like to treat fluids using classical continuum mechanics. We will describe a fluid as a continuum with various quantities that vary continuously at every point in time and space. These quantities are the pressure, temperature, mass density and flow velocity. The former three quantities are described by scalar fields while the latter is described by a vector field.

The canonical equations of fluid dynamics are derived from conservation laws. The first equation, known as the continuity equation for mass density is[1]

$$\frac{\partial}{\partial t}\varrho + \nabla \cdot (\varrho \boldsymbol{v}) = 0 \tag{1}$$

where ϱ is the density, $\boldsymbol{v}$ the flow velocity and t is time. Equation 1 is a statement of mass conservation in a differential volume element. It states that the time rate of change of mass density in a differential volume element is equal to the negative of the divergence of mass density flux. A similar continuity equation should be familiar from a number of other physics disciplines.

The second consititutive equation for fluid dynamics is a statement of conservation of momentum. The general Navier-Stokes equation is given by[2]

$$\varrho \left(\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \right) = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} + \boldsymbol{f}^R \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{f}$ is the body force, and $\boldsymbol{f}^R$ is a random force. Note that all terms are per unit volume. The Navier-Stokes equation is the equivalent of Newton's second law for fluid dynamics as can be seen on closer inspection. The first term $\varrho \frac{\partial}{\partial t}\boldsymbol{v}$ is the usual time dependent momentum term familiar from classical mechanics. The second term $\varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ is called the convective acceleration term because it depends on the spatial gradient of velocity rather than the time derivative. Notice that this term is zero if velocity is constant along the direction of flow. The convective acceleration term is non-linear making the Navier-Stokes equations impossible to solve in closed form. The right side of the equation contains the forces on the fluid. The divergence of the stress tensor represents the force on a fluid element from the surrounding fluid. The body force, $\boldsymbol{f}$, is any force on a fluid element; for example gravity. The random force, $\boldsymbol{f}^R$, accounts for thermal fluctuations in the fluid.

Finally, since we have three unknown quantities, we need a third equation in order to have a closed set. We can use the equation of state for an ideal gas in order to relate the pressure to the density:

$$p = \frac{\varrho k_B T}{m} \tag{3}$$

Here T is the temperature, k_B is the Boltzmann constant and m is the mass of a fluid particle.

Equations 1 and 2 will form the basis of our description of a viscous fluid. Other equations could be written reflecting the conservation of energy and particle number. In our discussion we will consider only isothermal systems in which there is only one particle species precluding the need for additional equations to describe the fluid.

ii. Newtonian Fluids

In order to close the set of Eqs. 1, 2, and 3 we need to specify the form of the stress tensor, $\boldsymbol{\sigma}$. It is here that we will define the notion of a Newtonian fluid. In general the stress tensor is composed of two pieces, the pressure tensor and the viscous stress tensor. The viscous stress tensor is related to the strain rate tensor by a viscosity tensor. A Newtonian fluid is one in which there is a linear relation between these stress and strain tensors, i.e. the viscosity tensor is constant and does not depend on strain rate. Newtonian fluids are isotropic and the viscosity tensor can be represented by the two constants, η and η^V. Accordingly, the stress tensor for a viscous, Newtonian fluid is defined in the following way[2]:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \eta\left[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\dagger\right] + \left[\eta^V - \frac{2}{3}\eta\right](\nabla\cdot\boldsymbol{v})\,\boldsymbol{I} \tag{4}$$

Here p is the pressure, $\boldsymbol{I}$ is the unit dyad, η is the shear viscosity, and η^V is the bulk viscosity. We can obtain the Navier-Stokes equation for a viscous, Newtonian fluid by inserting this stress tensor into Eq. 2 leaving

$$\varrho\left(\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla p + \eta\nabla^2\boldsymbol{v} + \left(\frac{\eta}{3} + \eta^V\right)\nabla(\nabla\cdot\boldsymbol{v}) + \boldsymbol{f} + \boldsymbol{f}^R. \tag{5}$$

Note that the convective acceleration term remains the only non-linear term in $\boldsymbol{v}$.

iii. The Reynolds Number

We now would like to assess the relative importance of the various terms in the Navier-Stokes equation. We can scale the velocity by a characteristic velocity v_0, the lengths by a characteristic length L_0, and the time by a characteristic time T_0 in order to obtain the dimensionless Navier-Stokes equation[2]:

$$Re_T\frac{\partial}{\partial t'}\boldsymbol{v}' + Re(\boldsymbol{v}'\cdot\nabla')\boldsymbol{v}' = -\nabla' p' + \nabla'^2\boldsymbol{v}' + \left(\frac{1}{3} + \frac{\eta^V}{\eta}\right)\nabla'(\nabla'\cdot\boldsymbol{v}') + \boldsymbol{f}' + \boldsymbol{f}^{R'} \tag{6}$$

Here we have introduced primed variables and operators that are all dimensionless and of order 1. The terms on the left hand side are known as the inertial terms while the terms on the right are known as the viscous terms.

Introducing the Reynolds number and the time dependent Reyolds number allows us to weigh the relative importance of the various terms. In terms of our characteristic parameters they are given by[2]

$$Re = \frac{\varrho v_0 L_0}{\eta} = \frac{v_0 L_0}{\nu}, \qquad Re_T = \frac{\varrho L_0^2}{\eta T_0} = \frac{L_0^2}{\nu T_0} \tag{7}$$

where we have introduced the kinematic viscosity $\nu = \eta/\varrho$. If one defines $T_0 = L_0/v_0$ then we have $Re_T = Re$ and the entire inertial side of the equation will be weighted by the single Reynolds number. However, it is useful to keep this difference in order to probe effects on different time scales.

We can now compute the Reynolds number for a model microswimmer. For our model we will take a sphere of radius $R = L_0 = 1\text{nm}$ in water $\eta = 10^{-3}\text{Pa s}$ moving with thermal velocity $v_0 = \sqrt{9k_BT/4\varrho\pi L_0^3}$. In this case we have

$$Re \approx 2\times 10^{-3}. \tag{8}$$

It is clear that, at this scale, we are working in the regime of low Reynolds number. We may then safely neglect the non-linear convective acceleration term in the Navier-Stokes equation leaving us with

$$\varrho \frac{\partial}{\partial t} \boldsymbol{v} = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \left(\frac{\eta}{3} + \eta^V \right) \nabla (\nabla \cdot \boldsymbol{v}) + \boldsymbol{f} + \boldsymbol{f}^{\boldsymbol{R}}. \tag{9}$$

Note that with the removal of the convective acceleration term we are now left with a *linear* Navier-Stokes equation. This is a great simplification of the original equation that will open up many approaches to dealing with the solution of the Navier-Stokes equation.

Finally, we can take the ratio of the time dependent Reynolds number to the unadorned one giving

$$\frac{Re_T}{Re} = \frac{1}{T_0} \frac{L_0}{v_0}. \tag{10}$$

Evidently we must keep the unsteady acceleration term on short time scales and may disregard it on long time scales, given that the Reynolds number is small.

II. Linear Fluctuating Hydrodynamics [2]

i. Assumptions and Properties

In the following we follow the derivation of R.G. Winkler in reference [2]. In order to understand how thermal fluctuations behave we will keep the unsteady acceleration term and assume only small fluctuations from stationary values, i.e.

$$\varrho = \rho + \delta \varrho, \qquad \boldsymbol{v} = \boldsymbol{0} + \delta \boldsymbol{v}, \qquad p = p + \delta p \tag{11}$$

We may then write the continuity and Navier-Stokes equations keeping only the first non-zero terms in density. This leaves us with

$$\frac{\partial}{\partial t} \delta \varrho + \rho \nabla \cdot \delta \boldsymbol{v} = 0 \tag{12}$$

for the continuity equation and

$$\rho \frac{\partial}{\partial t} \delta \boldsymbol{v} = -\nabla \delta p + \eta \nabla^2 \delta \boldsymbol{v} + \left(\frac{\eta}{3} + \eta^V \right) \nabla (\nabla \cdot \delta \boldsymbol{v}) + \boldsymbol{f} + \boldsymbol{f}^{\boldsymbol{R}} \tag{13}$$

for the Navier-Stokes equation. These equations are known as the linearised Landau-Lifshitz Navier-Stokes equations. We can then eliminate density by taking the divergence of Eq. 13 and using Eq. 3 leaving

$$\nabla \delta p - \frac{1}{c^2} \frac{\partial^2 \delta p}{\partial t^2} = \nabla \cdot (\eta \nabla^2 \delta \boldsymbol{v} + \left(\frac{\eta}{3} + \eta^V \right) \nabla (\nabla \cdot \delta \boldsymbol{v}) + \boldsymbol{f} + \boldsymbol{f}^{\boldsymbol{R}}) \tag{14}$$

where $c = \sqrt{k_B T / m}$ is the isothermal velocity of sound. Equations 13 and 14 are two equations for the two unknowns p and $\boldsymbol{v}$. Since they are linear with constant coefficients, they lend themselves to solution via Fourier transform.

We now need to know something about our random force in order to make progress. We define this random force in terms of a thermal stress tensor like so:

$$\boldsymbol{f}^R := \nabla \cdot \boldsymbol{\sigma}^R \tag{15}$$

The thermal stress tensor is defined to have the following moments[2]:

$$\langle \boldsymbol{\sigma}^R \rangle := 0 \tag{16}$$

$$\langle \sigma^R_{\alpha\beta}(\boldsymbol{r}, t) \sigma^R_{\alpha'\beta'}(\boldsymbol{r}', t') \rangle := 2 k_B T \eta_{\alpha\beta\alpha'\beta'} \delta(\boldsymbol{r} - \boldsymbol{r}') \delta(t - t') \tag{17}$$

Here η is the viscosity tensor for a Newtonian fluid, used implicitly earlier but defined as

$$\eta_{\alpha\beta\alpha'\beta'} = \eta (\delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\alpha\beta'} \delta_{\alpha'\beta}) - \left(\frac{2}{3} \eta - \eta^V \right) \delta_{\alpha\beta} \delta_{\alpha'\beta'}. \tag{18}$$

The thermal stress tensor is a stochastic process that depends on space and time. The process is Gaussian and Markovian. A Markovian process is one that is memoryless. Future states of the stochastic process depend only on the current state and not the past. Note especially that the mean value of the thermal tensor is zero.

Since the mean value of the thermal tensor is zero we also have

$$\langle \boldsymbol{f}^R \rangle = 0. \tag{19}$$

Accordingly, we may take the average of the entire Navier-Stokes equation to end up with

$$\rho \frac{\partial}{\partial t} \boldsymbol{v} = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \left(\frac{\eta}{3} + \eta^V \right) \nabla (\nabla \cdot \boldsymbol{v}) + \boldsymbol{f}. \tag{20}$$

This is just the usual Navier-Stokes equation with no change due to the fluctuating term. Apparently we must look to the second moment in order to learn more about how fluctuations behave in a fluid.

ii. The Long-Time Tail

As mentioned before, we can solve these linear equations by Fourier transform. In order to make progress we must then split the velocity Fourier components into transverse and longitudinal components. Note that this means transverse and longitudinal to $\boldsymbol{k}$ in the following way:

$$\hat{\boldsymbol{v}} \cdot \boldsymbol{k} = \hat{v}^L k, \qquad \hat{\boldsymbol{v}}^T \cdot \boldsymbol{k} = 0 \tag{21}$$

Taking the Fourier transform of Eqs. 13 and 14 and looking at the transverse component of velocity leaves us with

$$i\omega\rho\hat{\boldsymbol{v}}^T = i\boldsymbol{k}\hat{p} - \eta k^2 \hat{\boldsymbol{v}}^T + \hat{\boldsymbol{f}} + \hat{\boldsymbol{f}}^R \tag{22}$$

$$\left(\frac{\omega^2}{c^2} - k^2 \right) \hat{p} = i\boldsymbol{k} \cdot \left(\eta k^2 \hat{\boldsymbol{v}}^T - \hat{\boldsymbol{f}} - \hat{\boldsymbol{f}}^R \right) \tag{23}$$

Equipped with these equations, we may solve for the Fourier components of transverse velocity in terms of the body force and random force.

Now, since we have defined the correlation function for thermal stress tensor, the idea is to write the velocity correlation function in terms of the correlation function of the thermal force which can then be written in terms of the correlation function for the thermal stress tensor. For the transverse component of velocity we find the following autocorrelation function[2]:

$$\langle \hat{\boldsymbol{v}}^T(\boldsymbol{k}, t) \cdot \hat{\boldsymbol{v}}^T(-\boldsymbol{k}, 0) \rangle = \frac{2k_B T}{\rho V} e^{-\nu k^2 |t|} \tag{24}$$

We can then Fourier transform back to real space and take the average value over all displacements, i.e.

$$\langle \delta \boldsymbol{v}^T(t) \cdot \delta \boldsymbol{v}^T(0) \rangle = \sum_{\boldsymbol{k}} \langle \hat{\boldsymbol{v}}^T(\boldsymbol{k}, t) \cdot \hat{\boldsymbol{v}}^T(-\boldsymbol{k}, 0) \rangle \langle e^{-i\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \rangle. \tag{25}$$

It we assume that the displacements are distributed normally we can evaluate this sum. In the limit of an infinite system the sum reduces to an integral that we can solve leaving us with[2]

$$\langle \delta \boldsymbol{v}^T(t) \cdot \delta \boldsymbol{v}^T(0) \rangle = \frac{k_B T}{4\rho} \frac{1}{(\pi(\nu + D)t)^{3/2}} \tag{26}$$

where D is the diffusion coefficient. Here we can clearly see what is known as the long-time tail in hydrodynamics. The velocity correlation function decays by $t^{-3/2}$. This is a marked qualitative difference from most equilibrium systems. When we have a usual exponential decay, we may define a characteristic time for the system and divide phenomena into short and long time scales. It is not as clear what is meant by short and long time scales when we have algebraic decay.

III. The Creeping Motion Equations

i. The Equations

Now that we have taken an excursion to understand how fluctuations behave in a fluid, we want to continue to modify the Navier-Stokes equation. We begin again now with the average Navier-Stokes equation given by Eq. 20. We now, however, consider longer time scales so that $Re_T \to 0$. This leaves us with

$$-\nabla p + \eta \nabla^2 \boldsymbol{v} + \left(\frac{\eta}{3} + \eta^V\right) \nabla(\nabla \cdot \boldsymbol{v}) + \boldsymbol{f} = 0 \tag{27}$$

We can make further simplifications if we assume incompressiblility of fluid. This means that the density of a fluid element is constant. If we make this assumption then the continuity equation given by Eq. 12 reads[1]

$$\nabla \cdot \boldsymbol{v} = 0. \tag{28}$$

This is the equation of continuity for an incompressible fluid and will be the continuity equation we use for the remainder of this review.

Evidently this leads to a great simplification in the Eq. 27 leaving us with

$$-\nabla p + \eta \nabla^2 \boldsymbol{v} + \boldsymbol{f} = 0. \tag{29}$$

This ultimate equation is known as the steady-state Stokes equation. This equation together with our newly simplified continuity equation are known as the creeping motion equations and are the central figure of hydrodynamics at low Reynolds number.

ii. Boundary Conditions

The solution of Eq. 29 depends on boundary conditions which we discuss in the following. If a fluid is in contact with a solid wall, it has been shown experimentally that velocity of the fluid is the same as that of the wall. This is known as the no-slip condition and leads to[1]

$$\boldsymbol{v}_{\text{fluid}} = \boldsymbol{v}_{\text{wall}} \tag{30}$$

where the velocities are evaluated at the wall.

Similarly, if two fluids form a boundary, continuity of both the stress tensor and the velocity field across the boundary is required. This leads to the following boundary conditions for a fluid-fluid interface[6]:

$$\boldsymbol{v}_{\text{fluid1}} = \boldsymbol{v}_{\text{fluid2}} \tag{31}$$

$$\boldsymbol{\sigma}_{\text{fluid1}} = \boldsymbol{\sigma}_{\text{fluid2}} \tag{32}$$

Here the velocity and stress tensor are again evaluated at the boundary. Of course, if you are working in an unbounded fluid then the velocity should be defined at infinity.

IV. Methods for Solution

i. Variational Method for Energy Dissipation

We are working with viscous fluids. This means that there is some friction and resistance to motion and this resistance causes energy to dissipate as heat. If we consider a small fluid element moving with velocity, $\boldsymbol{v}$, we can find the total rate of work done on the element by[1]

$$P = \frac{dW}{dt} = \int_S \boldsymbol{v} \cdot \boldsymbol{\sigma} \cdot d\boldsymbol{S} = \int_V \nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}) dV \tag{33}$$

where we have used Gauss' divergence theorem to find the second form. We may then use the identity

$$\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}) = \boldsymbol{\sigma} : \nabla \boldsymbol{v} + \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{\sigma}) \tag{34}$$

to write the power as

$$P = \int_V [\boldsymbol{\sigma} : \nabla \boldsymbol{v} + \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{\sigma})] dV = \int_V [-p(\nabla \cdot \boldsymbol{v}) + \eta [\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\dagger] : \nabla \boldsymbol{v} + \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{\sigma})] dV \tag{35}$$

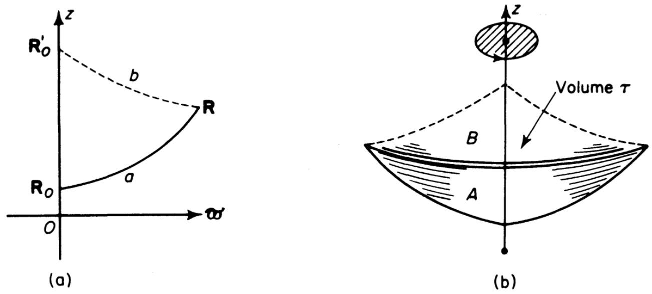

Figure 1: *Images showing the definition of the stream function.[1]*

where we have inserted the Newtonian stress tensor to decompose the first term. The first term on the right hand side is the rate of work done in changing the volume of a fluid element and is zero for an incompressible fluid. The third term is the rate of work done in moving the entire fluid element. The second term corresponds to the rate of work done in deforming the fluid element. This second term is the one that corresponds to energy dissipation. We can write this local energy dissipation rate more compactly as[1]

$$\Phi = 2\eta \mathbf{\Delta} : \mathbf{\Delta} \tag{36}$$

where

$$\mathbf{\Delta} = \frac{1}{2}\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\dagger}\right] \tag{37}$$

is the rate of strain tensor for an incompressible, Newtonian fluid. The total energy dissipation rate over a volume is then given by the integral

$$E = \int_V \Phi dV. \tag{38}$$

It can be shown that creeping flow minimizes energy dissipation. Therefore, if one finds a velocity field that satisfies the boundary conditions and the continuity equation and that velocity field minimizes the energy dissipation rate then that velocity field must satisfy the steady state Stokes equation. Evidently, a variational method may then be used to find a velocity field that satisfies the steady state Stokes equation as long as we know the velocity on some bounding surface.

ii. The Stream Function

When the system under consideration has what is known as axial symmetry, the system can effectively be described two dimensionally. In two dimensions, an incompressible flow system can be described by a single scalar function known as the stream function. This greatly simplifys the problem since solving for a scalar function is much easier than solving for a vector function, i.e. the velocity field. We follow the derivation of Happel and Brenner in chapter 4 of their book. There are many symmetric three dimensional systems that can be modeled by a stream function but we will focus on axial symmetry. Axial symmetry can be realized in flow past a body of revolution, a sphere for instance, along its symmetry axis. In fact we will use the problem of flow past a sphere as our example to illustrate the stream function. Axial symmetry is defined mathematically as

$$\frac{\partial \boldsymbol{v}}{\partial \phi} = \mathbf{0}, \qquad \boldsymbol{i}_{\phi} \cdot \boldsymbol{v} = 0 \tag{39}$$

where ϕ is the azimuthal angle about the axis of symmetry. The former means that the velocity does not depend on the azimuthal angle and the latter says that the flow velocity has no component along the azimuthal direction. The fluid can then be defined completely in a meridian plane. We will work initially in cylindrical coordinates.

The stream function is then motivated and then defined as follows. As in figure 1, imagine two curves, a and b, connecting some point $\boldsymbol{R}$ to two different points $\boldsymbol{R}_0$ and $\boldsymbol{R}_0'$ lying on the axis of symmetry. We can then imagine creating surfaces A and B by rotating the synonymous curves around the axis of symmetry. We then label the volume enclosed by this surface as τ and the flow rate through A as Q_A and through B as Q_B. If we consider an incompressible fluid we have $\nabla \cdot \boldsymbol{v} = 0$ and it is clear that no fluid can be created or destroyed in τ so that $Q_A = Q_B = Q$. Q is then only a function of $\boldsymbol{R}$ and time so that we can define the stream function as [1]

$$\Psi(\mathbf{R}, t) = \frac{Q}{2\pi}. \tag{40}$$

We also define the stream function to be zero along the axis of symmetry.

We now would like to find a way to relate our scalar stream function back to the vector field of interest, namely the velocity. We can equate the flow rate through a surface created by a curve ending at $\boldsymbol{R}$, less 2π, as in our definition of the stream function(see fig. 2 *a*) to the integral of $d\Psi$ along the same curve(see fig. 2 *b*), i.e.[1]

$$\Psi = \frac{1}{2\pi}\int_S \boldsymbol{v}\cdot\boldsymbol{n}\, dS = \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} \boldsymbol{v}\cdot\boldsymbol{n}\varpi\,|d\boldsymbol{R}| \tag{41}$$

and

$$\Psi = \int_{\boldsymbol{R}_O}^{\boldsymbol{R}} d\Psi = \int_{\boldsymbol{R}_O}^{\boldsymbol{R}} d\boldsymbol{R}\cdot\nabla\Psi = \int_{\boldsymbol{R}_O}^{\boldsymbol{R}} \boldsymbol{t}\cdot\nabla\Psi|d\boldsymbol{R}|. \tag{42}$$

We can then equate the integrands leaving us with

$$\boldsymbol{n}\cdot\boldsymbol{v}\varpi - \boldsymbol{t}\cdot\nabla\Psi = 0. \tag{43}$$

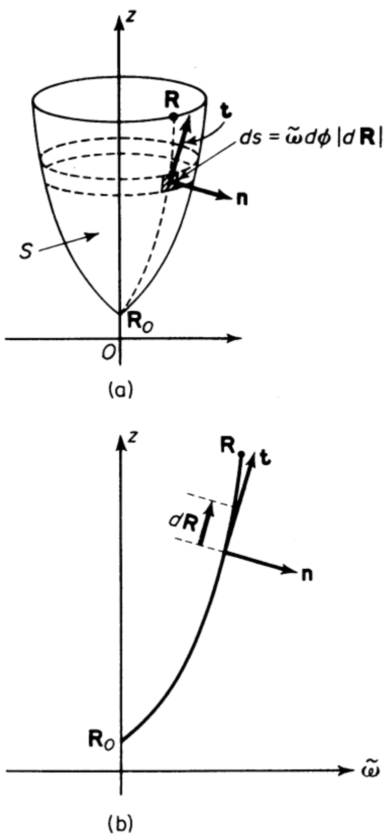

Figure 2: *Geometric considerations for relating velocity field to stream function [1]*

Our coordinate system is right-handed and orthogonal in that $\boldsymbol{t} = \boldsymbol{n}\times\boldsymbol{i}_\phi$ so that we may write

$$\boldsymbol{n}\cdot(\boldsymbol{v}\varpi - \boldsymbol{i}_\phi\times\nabla\Phi). \tag{44}$$

Finally, since the direction of $\boldsymbol{n}$ is arbitrary we find that the expression in parentheses must vanish. This leaves us with the following result:

$$\boldsymbol{v} = \frac{1}{\varpi}\boldsymbol{i}_\phi\times\nabla\Psi = -\nabla\times\left(\boldsymbol{i}_\phi\frac{\Psi}{\varpi}\right) \tag{45}$$

The velocity components in spherical coordinates which will be useful in a moment are

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \qquad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \tag{46}$$

We now turn our attention to the problem of a sphere of radius a moving with velocity U in the z direction. In order to solve this problem we need a differential equation to solve for the stream function. To do this we take the curl of the unforced Stokes equation leaving[1]

$$\nabla\times(\nabla^2\boldsymbol{v}) = \mathbf{0} \tag{47}$$

because the curl annihilates the gradient of a scalar. We can then express this equation in terms of the stream function by plugging in $\boldsymbol{v}(\Psi)$ leaving us with

$$E^4\Psi = 0 \tag{48}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{49}$$

in spherical coordinates.

Now we collect the boundary conditions relevant to the translating sphere problem. We write the velocity on the boundary of the sphere in spherical coordinates as

$$v_r(a) = U\cos\theta, \qquad v_\theta(a) = -U\sin\theta, \qquad \boldsymbol{v}(r\to\infty) = \mathbf{0} \tag{50}$$

With these along with eq 46 we can find the boundary conditions for the stream function:

$$\Psi|_{r=a} = -\frac{1}{2}Ua^2\sin^2\theta, \qquad \left.\frac{\partial\Psi}{\partial r}\right|_{r=a} = -Ua\sin^2\theta, \qquad \frac{\Psi(r\to\infty)}{r^2} = 0 \tag{51}$$

We can now solve for the flow of a translating sphere. The solution to the differential equation before considering boundary conditions is[1]

$$\Psi = \sin^2\theta\left(\frac{1}{10}Ar^2 - \frac{1}{2}Br + Cr^2 + \frac{D}{r}\right). \tag{52}$$

The boundary conditions at infinity imply that $A = C = 0$ and the conditions on the sphere imply that

$$B = \frac{3}{2}Ua, \quad D = \frac{1}{4}Ua^2 \tag{53}$$

leaving us, ultimately, with

$$\Psi = \frac{1}{4}Ua^2\sin^2\theta\left(\frac{a}{r} - 3\frac{r}{a}\right). \tag{54}$$

Finally, we can use Eq. 46 to find the velocity field leaving us with

$$v_r = -\frac{1}{2}U\cos\theta\left(\frac{a}{r}\right)^2\left(\frac{a}{r} - 3\frac{r}{a}\right) \quad \text{and} \quad v_\theta = -\frac{1}{4}U\sin\theta\left(\frac{a}{r}\right)\left[\left(\frac{a}{r}\right)^2 - 3\right]. \tag{55}$$

Note especially that the velocity is proportional to a term in $1/r$ and a term in $1/r^3$. We can then use the Stokes equation to solve for the pressure field in order to completely describe the fluid.

Now that we know the flow we would like to know the force on the translating sphere. Because the force on a surface element is given by $\boldsymbol{\sigma}\cdot d\boldsymbol{S}$ we have the total force and torque on the body as[1]

$$\mathbf{F} = \int_{\text{body}} \boldsymbol{\sigma}\cdot d\mathbf{S} \quad \text{and} \quad \boldsymbol{T} = \int_{\text{body}} \boldsymbol{r}\times\boldsymbol{\sigma}\cdot d\boldsymbol{S} \tag{56}$$

where the torque is computed from an arbitrary origin. In the case of our sphere we have

$$F_z = -6\pi\eta a U. \tag{57}$$

This is known as the Stokes law and is a celebrated result in hydrodynamics. It was derived by Stokes by a different method.

iii. Singularities

The final method of solution we will address is known as the method of singularities. Recalling that the Stokes equations (Eqs. 28 and 29) are linear, it is clear that flow can be modelled as a linear superposition of flows. The solution of a differential equation due to an impulse (represented by a Dirac-delta function) is known as a fundamental solution or Green's function. In the study of hydrodynamics these fundamental solutions are called singularities. We can then represent the total flow as a superposition of the flows due to these singularities.

The most basic singularity, due to a point force solves

$$-\nabla p + \eta\Delta\boldsymbol{v} + f\hat{\boldsymbol{e}}\delta(\boldsymbol{r}-\boldsymbol{r}_0) = 0, \qquad \nabla\cdot\boldsymbol{v} = 0 \tag{58}$$

which is just the Stokes equations with a point force. The solution to this equation is known as a Stokeslet and is given by[5]

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{f}{8\pi\eta}\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}) \tag{59}$$

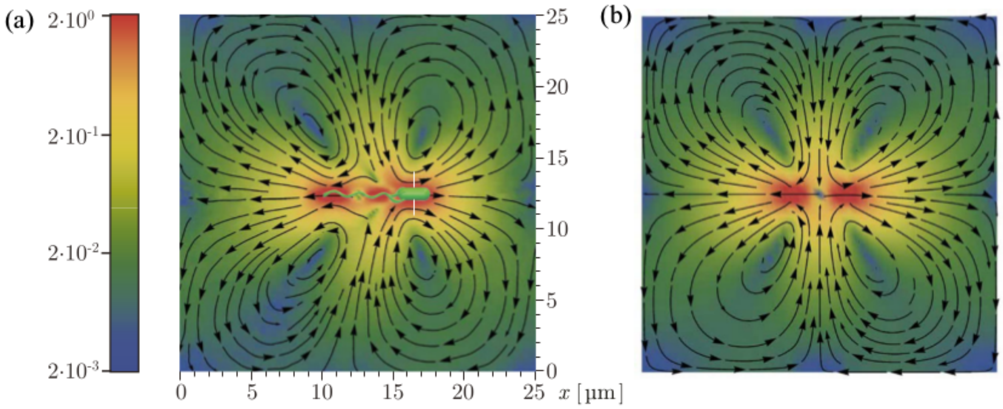

Figure 3: *(a) The flow field due to a swimmer with a helical tail simulated using multi-particle collision dynamics. (b) The flow field due to a force dipole. [2]*

where

$$\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}) = \frac{1}{R}\left(\hat{\boldsymbol{e}} + \frac{[\hat{\boldsymbol{e}}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)](\boldsymbol{r}-\boldsymbol{r}_0)}{R^2}\right). \tag{60}$$

Here $\hat{\boldsymbol{e}}$ is the unit vector corresponding to the point force and $R = |\boldsymbol{r}-\boldsymbol{r}_0|$. Notice that the Stokeslet decays according to $1/R$.

Higher order singularities may be found by taking derivatives of a lower order singularity. The force dipole and force quadrupole are given by

$$\boldsymbol{G_D}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{d}},\hat{\boldsymbol{e}}) = \hat{\boldsymbol{d}}\cdot\nabla_0\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}) \tag{61}$$

and

$$\boldsymbol{G_Q}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{c}},\hat{\boldsymbol{d}},\hat{\boldsymbol{e}}) = \hat{\boldsymbol{c}}\cdot\nabla_0\boldsymbol{G_D}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{d}},\hat{\boldsymbol{e}}) \tag{62}$$

where the gradient acts on $\boldsymbol{r}_0$. Each successive derivative causes the singularity to decay with a higher power of R so that in the far field lower order singularities will dominate.

Another set of singularity solutions can be found for the case that $p = 0$ in the Stokes equation. In this case, Stokes equation becomes the Laplace equation and the flow is irrotational, i.e. $\nabla\times\boldsymbol{v} = 0$. These solutions are called potential flow solutions because the velocity may be written as the gradient of a scalar. The lowest order singularity of this class is due to a point source and is given by[5]

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{M}{4\pi}\boldsymbol{U}(\boldsymbol{r}-\boldsymbol{r}_0) \tag{63}$$

where

$$\boldsymbol{U}(\boldsymbol{r}-\boldsymbol{r}_0) = \frac{\boldsymbol{r}-\boldsymbol{r}_0}{R^3}. \tag{64}$$

Note that the point source singularity decays as $1/R^2$. The source dipole and quadrupole are again given by differentiation:

$$\boldsymbol{D}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}) = \hat{\boldsymbol{e}}\cdot\nabla_0\boldsymbol{U}(\boldsymbol{r}-\boldsymbol{r}_0), \qquad \boldsymbol{Q}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{d}},\hat{\boldsymbol{e}}) = \hat{\boldsymbol{d}}\cdot\nabla_0\boldsymbol{D}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}) \tag{65}$$

Here again subsequent differentiation leads to quicker decay. Note also that we can relate source singularities to force singularities via

$$\boldsymbol{D}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}) = -\frac{1}{2}\nabla_0^2\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}_0;\hat{\boldsymbol{e}}). \tag{66}$$

These two sets of singularites form a complete set of solutions to the creeping motion equations.

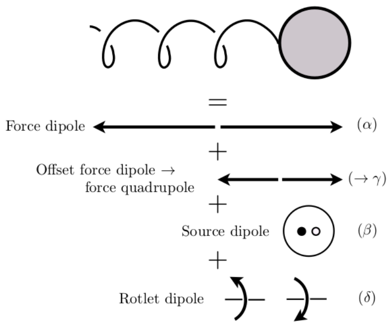

Figure 4: *The far-field expansion of a microswimmer. A swimmer is represented by four different singularities[5]*

In analogy with the multipole expansion in electrostatics, a flow may be expanded and represented by a finite number of singularities. For example, a swimmer with a helical tail may be modelled by a force dipole, a force quadrupole, a source dipole, and a rotlet dipole as seen in figure 4. The first decays as $1/R^2$, while the rest decay as $1/R^3$[5]. A rotlet is simply a combination of force dipoles and represents a point torque. We could also model the translating sphere we looked at in the previous section. This flow is given by the superposition of a Stokeslet and a source dipole located at the center of the sphere which decay as $1/R$ and $1/R^3$, respectively[3]. Recall that the velocity field had two terms: One that depends on $1/R$ and one that depends on $1/R^3$.

In the far-field, the slowest decaying singularity dominates so that we may approximate this helical tailed swimmer by a force dipole. Figure 3 shows the flow field from a force dipole juxtaposed with the flow field generated by the same swimmer simulated using multi-particle collision dynamics. The qualitative similarity may be seen from this figure.

A complication arrises when flow singularities are considered in a system with a solid boundary, for instance an infinite plane. Clearly the no-slip boundary condition will not be met in general. Again in analogy with the method of images of electrostatics, we can use a superposition of the original flow plus a collection of singularities on the opposite side of the wall that will cause the no-slip condition to be met on the wall. For example, the image system required to meet the no-slip condition due to a Stokeslet is the combination of a stokeslet, a force dipole, and a source dipole[4]. This method of images can be used to model the behavior of swimmers near a boundary.

V. Concluding Remarks

Over the course of this overview we began with the mathematical description of a fluid using classical continuum mechanics and introduced the continuity and Navier-Stokes equations. We then introduced the stress tensor for a viscous, Newtonian fluid and introduced the Reynolds number in order to study flow on various scales. Flow on the scale relevant to active matter was revealed to be represented by low Reynolds numbers. We then analyzed the effect due to a fluctuating thermal force and found that the correlation function of transverse velocity decays algebraically. Finally, we introduced the concept of an incompressible fluid leaving us with the steady state Stokes equations. We then discussed a few ways to solve these linear equations, namely the variational principle, the stream function, and the method of singularities.

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